

ABSENCE OF BOUND STATES FOR WAVEGUIDES IN 2D PERIODIC STRUCTURES

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ABSTRACT. We study a Helmholtz-type spectral problem in a two-dimensional medium consisting of a fully periodic background structure and a perturbation in form of a line defect. The defect is aligned along one of the coordinate axes, periodic in that direction (with the same periodicity as the background) and bounded in the other direction. This setting models a so-called “soft-wall” waveguide problem. We show that as a consequence of the perturbation, no eigenvalues can arise in the band gaps of the unperturbed, fully periodic problem.

1. INTRODUCTION

Consider a Helmholtz-type spectral problem on \mathbb{R}^2 of the form

$$(1) \quad -\Delta u = \lambda \varepsilon u.$$

$\varepsilon \in L^\infty(\mathbb{R}^2, \mathbb{R})$ is given by

$$\varepsilon = \varepsilon_0 + \varepsilon_1$$

where ε_0 is periodic with respect to the lattice \mathbb{Z}^2 . ε_1 is periodic in x_2 direction (with respect to \mathbb{Z}), $\text{supp } \varepsilon_1 \subset (0, 1) \times \mathbb{R}$. ε and ε_0 are bounded from below by a positive constant. The physical interpretation is as follows: the spectral problem (1) models the propagation of polarized electromagnetic waves in a periodic medium, described by ε_0 , perturbed by a straight waveguide (see figure 1).

The unperturbed spectral problem (i.e., ε replaced by ε_0 in (1)) is periodic with respect to \mathbb{Z}^2 and its spectrum has the well-known *band gap* structure. Note that the problem (1) is no longer periodic with respect to the full lattice \mathbb{Z}^2 . The perturbation may induce additional spectrum in a band gap of the periodic background operator (see the literature references below). If present, this spectrum should correspond to guided modes propagating in the direction of the waveguide. However, in order to associate the additional spectrum with truly guided modes, one has to prove that no eigenvalues of (1) are contained in the band gaps. Such an eigenvalue corresponds to a localized mode (bound state) on the whole space and its existence should be highly unlikely, according to physical intuition, though in general very hard to prove mathematically. This is the question we would like to address in this paper. Our main result guarantees the absence of such eigenvalues for the problem (1), as long as λ lies in a band gap of the unperturbed operator.

Problems of a related nature have been a subject of intensive study for some decades: while the absence of singular continuous spectrum for periodic operators

can be proven rather easily, the challenging task for the periodic operators of mathematical physics is to exclude the existence of point spectrum.

For the Schrödinger operator with a potential periodic in all space directions, the absolute continuity of the spectrum was proven in the celebrated paper by L. Thomas [18]. His results were extended to Schrödinger operators with magnetic potentials, by M. Sh. Birman and T. Suslina in [2] and by A. Sobolev [16]. An overview on results and open problems related to absolute continuity for periodic problems is given in the papers [9], [13] and [17]. The study of periodic waveguides goes back to [3]. The problem of absolute continuity of the spectrum in periodic waveguides with “hard walls” (i.e. where the guided mode is confined by e.g. Dirichlet boundary conditions) has been considered in [15], [5] and more recently, in [6].

Although very relevant for modern developments in optics, there is definitely a lack of mathematical publications dealing with “soft wall” waveguides, i.e., where the guided modes are allowed to penetrate the surrounding medium with an exponential decay ([10], [11]). Sufficient conditions for the existence of spectrum of (1) in band gaps of the periodic background have been derived by Ammari and Santosa in [1] and by P. Kuchment and B. Ong in [11], [12]. The paper [4] by N. Filonov and F. Klopp treats a different type of “soft-wall” waveguide problem, namely a periodic waveguide surrounded by a medium which is asymptotically homogeneous in lateral direction.

To the authors’ knowledge, the present paper contributes the first result on nonexistence of bound states in periodic waveguides which are embedded into a periodic background structure. The problem is highly nontrivial, since the standard Thomas approach is not applicable in its usual form (see e.g. the discussion in P. Kuchment’s review article [12]). The main difficulty comes precisely from the “soft wall” property of the problem.

We still use a version of Thomas’ idea of considering complex quasimomenta with large imaginary part. However, due to lack of compactness, several new techniques are needed, as well as careful estimates on the fully periodic operator. We refer to section 3 for an overview. The restriction to a two-dimensional situation is crucial for our approach. The corresponding problem in higher space dimensions remains open.

2. NOTATION AND PRELIMINARIES.

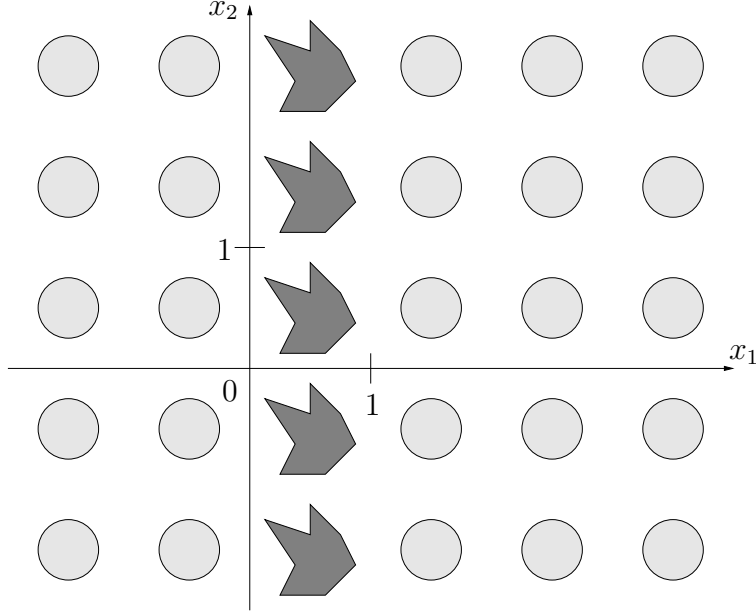
We introduce some notation that will be used below. Let $S := \mathbb{R} \times (0, 1)$ be the strip and $\Omega = (0, 1)^2$ the unit cell. Bold letters will indicate vectors, for example $\mathbf{k} = (k_1, k_2)$, $\mathbf{m} = (m_1, m_2)$, $\boldsymbol{\eta} = (\eta_1, \eta_2)$. All operator norms will be denoted by $\|\cdot\|$, since it will be clear from the context on which spaces the operator acts in each case.

Let $\varepsilon_0, \varepsilon_1 \in L^\infty(\mathbb{R}^2, \mathbb{R})$. ε_0 is assumed to be periodic with respect to \mathbb{Z}^2 , whereas for ε_1 we assume

$$\varepsilon_1(x_1, x_2 + m) = \varepsilon_1(x_1, x_2) \quad (m \in \mathbb{Z})$$

and $\text{supp } \varepsilon_1 \subset (0, 1) \times \mathbb{R}$. Both ε_0 and $\varepsilon := \varepsilon_0 + \varepsilon_1$ shall be bounded from below by positive constants, and moreover we assume that there exists a nonempty open set

FIGURE 1. Illustration of the periodic waveguide.



\mathcal{M} with

$$\inf_{\mathcal{M}} |\varepsilon_1| > 0.$$

$H_{\text{per}}^k(S)$ denotes the Sobolev space of functions periodic in x_2 -direction, and $H_{\text{per}}^k(\Omega)$ denotes the Sobolev space of periodic functions on the unit cell.

We will need “shifted” Laplacian operators on S and on Ω . For $k_2 \in \mathbb{C}$, $-\Delta_{k_2}$ will denote the operator

$$-\Delta_{k_2} := -(\nabla + i(0, k_2)) \cdot (\nabla + i(0, k_2))$$

acting on functions in $H_{\text{per}}^2(S)$. For $\mathbf{k} \in \mathbb{C}^2$, $-\Delta_{\mathbf{k}}$ denotes

$$-\Delta_{\mathbf{k}} := -(\nabla + i\mathbf{k}) \cdot (\nabla + i\mathbf{k})$$

with $H_{\text{per}}^2(\Omega)$ as domain.

The Floquet-Bloch transform in x_2 direction (see [8], [9])

$$(Vf)(x_1, x_2, k_2) := \frac{1}{\sqrt{2\pi}} \sum_{n \in \mathbb{Z}} e^{ik_2(n-x_2)} f(x_1, x_2 - n)$$

maps $L^2(\mathbb{R}^2)$ isometrically onto $L^2(S \times (-\pi, \pi))$. It is well-known that V can be used to reduce the spectral problem (1) to the strip S ; namely, $-\frac{1}{\varepsilon}\Delta$ can be expressed as a direct integral

$$-\frac{1}{\varepsilon}\Delta = \int_{[-\pi, \pi)}^{\oplus} -\frac{1}{\varepsilon}\Delta_{k_2} dk_2$$

and as a consequence, the spectrum of the self-adjoint operator $-\frac{1}{\varepsilon}\Delta$ is decomposed into the union of the spectra of problems on the strip:

$$\sigma\left(-\frac{1}{\varepsilon}\Delta\right) = \overline{\bigcup_{k_2 \in [-\pi, \pi]} \sigma\left(-\frac{1}{\varepsilon}\Delta_{k_2}\right)}.$$

Using the full periodicity, on the other hand, the spectrum of the periodic operator $-\frac{1}{\varepsilon_0}\Delta$ is decomposed according to

$$\sigma\left(-\frac{1}{\varepsilon_0}\Delta\right) = \bigcup_{\mathbf{k} \in [-\pi, \pi]^2} \sigma\left(-\frac{1}{\varepsilon_0}\Delta_{\mathbf{k}}\right).$$

For $\lambda \notin \sigma\left(-\frac{1}{\varepsilon_0}\Delta\right)$ the inverse operator of $(-\Delta_{\mathbf{k}} - \lambda\varepsilon_0)$ will frequently appear below, and we write

$$T(\mathbf{k}) = T(k_1, k_2) := \frac{1}{2\pi}(-\Delta_{\mathbf{k}} - \lambda\varepsilon_0)^{-1}$$

whenever $(-\Delta_{\mathbf{k}} - \lambda\varepsilon_0)^{-1}$ exists. This can be true or not, depending on $\mathbf{k} \in \mathbb{C}^2$. Furthermore, we often consider $T(k_1, k_2)$ as a function of k_1 for fixed k_2 . If for fixed $k_2 \in \mathbb{C}$, $T(k_1, k_2)$ exists for some $k_1 \in \mathbb{C}$, then $k_1 \mapsto T(k_1, k_2)$ is meromorphic.

For $f \in L^2(\Omega)$, we define $\tilde{f} \in L^2(S)$ to be

$$\tilde{f} = f \text{ on } \Omega, \quad \tilde{f} = 0 \text{ elsewhere.}$$

An important role is played by $(-\Delta_{k_2} - \lambda\varepsilon_0)^{-1}\tilde{f}$. It is straightforward to apply the Floquet-Bloch reduction in x_1 -direction to show the following formula:

$$(2) \quad ((-\Delta_{k_2} - \lambda\varepsilon_0)^{-1}\tilde{f})(\mathbf{x}) = \int_{-\pi}^{\pi} e^{ik_1 x_1} (T(k_1, k_2) e^{-ik_1 \cdot} f)(\mathbf{x}) dk_1$$

($\mathbf{x} \in \Omega$). Here $e^{-ik_1 \cdot}$ means the function $(y_1, y_2) \mapsto e^{-ik_1 y_1}$. It is convenient to interpret the integral in (2) as a Bochner integral with values in $L^2(\Omega)$.

As usual, it will be important to diagonalize $-\Delta_{\mathbf{k}}$ using Fourier series on Ω . Any $u \in H_{\text{per}}^2(\Omega)$ can be expanded into Fourier modes $\{e^{i\mathbf{m} \cdot \mathbf{x}}\}_{\mathbf{m} \in 2\pi\mathbb{Z}^2}$; on the level of Fourier coefficients, the action of $-\Delta_{\mathbf{k}}$ on u is given by multiplication with

$$s(\mathbf{m}, \mathbf{k}) = (\mathbf{m} + \mathbf{k})^2.$$

Remark 2.1. *We will need the following facts from the perturbation theory of linear operators (see [7]):*

- (i) *The set \mathcal{D} of \mathbf{k} where $T(\mathbf{k})$ exists as a bounded operator in $L^2(\Omega)$ is a open subset of \mathbb{C}^2 ;*
- (ii) *the mapping $\mathbf{k} \mapsto T(\mathbf{k})$ is analytic on \mathcal{D} .*

3. MAIN RESULT AND GENERAL PLAN OF THE PAPER.

3.1. The main result.

Theorem 3.1. *Let $\lambda \notin \sigma(-\frac{1}{\varepsilon_0}\Delta)$. In $H^2(\mathbb{R}^2)$, the equation (1) has only the trivial solution.*

We now sketch our approach to the proof of theorem 3.1.

1. The first step is to use the usual Floquet-Bloch reduction in x_2 -direction. Applied to (1), this yields a problem on the strip S . Thus, the existence of a nontrivial solution of (1) implies that

$$(3) \quad (-\Delta_{k_2} - \lambda\varepsilon)u = 0, \quad u \in H_{\text{per}}^2(S)$$

has a nontrivial solution for almost all $k_2 \in \mathbb{R}$. It is not straightforward to apply Thomas' idea (extension to complex k_2) directly to (3), since S is unbounded and hence the spectrum of the strip problem is not discrete. First observe that since λ lies in a band gap of the unperturbed system, $(-\Delta_{k_2} - \lambda\varepsilon_0)$ is invertible for all real k_2 . By making use of this fact, we derive an equivalent analytic Fredholm-type equation for the new unknown function $v \in L^2(\Omega)$:

$$(4) \quad v + \lambda G(k_2)v = 0$$

with some compact operator $G(k_2) : L^2(\Omega) \rightarrow L^2(\Omega)$ to be introduced below in (6). $G(k_2)$ is defined for k_2 in a neighborhood of the real axis.

2. In the second step, we construct an analytic continuation of the operator family $G(k_2)$ to values k_2 with large imaginary part. To this end, we exploit the periodicity of the unperturbed operator in x_1 direction, via the formula (2). Basically, the idea consists in deforming the integral in (2) into the complex plane, thereby obtaining a representation involving an integral over a line lying sufficiently far away from the real axis (in the k_1 -plane) plus a sum over the residues of the meromorphic operator-valued function $T(\cdot, k_2)$. Since in the following we want to let $\text{Im } k_2 \rightarrow \infty$ (similar to the Thomas approach), it is therefore crucial to understand the movement of the poles of $T(\cdot, k_2)$ as k_2 varies. In general, however, the poles have algebraic singularities as functions of k_2 ; in order to overcome this difficulty, we construct an analytic continuation only in the neighborhood of a certain path in the complex plane, carefully avoiding the algebraic branching points. Note here that results from complex analysis of a single variable enter. So the restriction to two space dimensions is important, since then we have to deal with only two complex quasimomenta.
3. In the third and technically most difficult step, we study the behavior of the analytically continued operator-valued family for values k_2 with large imaginary part. By carefully estimating the symbol of the shifted cell Laplacian $-\Delta_{\mathbf{k}}$, we will be able to localize the poles of $T(\cdot, k_2)$ for $\text{Im } k_2$ large. The essential technical estimates are contained in theorem 8.4, and here the two-dimensionality of the problem is required, too. Finally, these estimates allow us to conclude by a Neumann series argument that (4) has only the trivial solution.

3.2. Plan of the paper. The remainder of the paper is structured as follows: in section 4, we give the analytic Fredholm equation involving the operator $G(k_2)$. Then, in section 5 we describe in detail the analytic continuation process. The

study of the continued operator family for large imaginary values of k_2 occupies section 6. Finally, section 7 contains the proof of the main result. Throughout the paper, we will refer to results from the appendix, which also contains the major bulk of technical computations.

4. REFORMULATION OF THE PROBLEM

Fix $\lambda \notin \sigma\left(-\frac{1}{\varepsilon_0}\Delta\right)$, let $k_2 \in \mathbb{R}$ and suppose $u \in H_{\text{per}}^2(S)$ solves

$$(5) \quad -\Delta_{k_2}u - \lambda(\varepsilon_0 + \varepsilon_1)u = 0 \text{ on } S.$$

Moreover, let $G(k_2) : L^2(\Omega) \rightarrow L^2(\Omega)$ be defined by

$$(6) \quad G(k_2)v := \varepsilon_1(-\Delta_{k_2} - \lambda\varepsilon_0)^{-1}\tilde{v}$$

where $\tilde{v} = v$ on Ω and $\tilde{v} = 0$ outside.

Lemma 4.1. *If $k_2 \in \mathbb{R}$ and $u \neq 0$ solves (5), then $v \in L^2(\Omega)$ defined by $v = \varepsilon_1 u$ solves*

$$(7) \quad v + \lambda G(k_2)v = 0 \quad \text{on } \Omega$$

and $v \neq 0$.

Proof. First we prove that $v \neq 0$; thus, assume the contrary. Since $|\varepsilon_1|$ is bounded away from zero on a nonempty open set, this implies $u = 0$ on a such a set. Using the fact that u solves (5) and a unique continuation principle (see [14]), we conclude $u \equiv 0$ on the whole of S , a contradiction.

Since $\lambda \notin \sigma\left(-\frac{1}{\varepsilon_0}\Delta\right)$, $(-\Delta_{k_2} - \lambda\varepsilon_0)^{-1}$ exists as a bounded operator in $L^2(S)$. Hence from $(-\Delta_{k_2} - \lambda\varepsilon)u = 0$ we get

$$0 = u + \lambda(-\Delta_{k_2} - \lambda\varepsilon_0)^{-1}\varepsilon_1 u.$$

Multiplying the equation with ε_1 , writing $v = \varepsilon_1 u$ and using (6), we get the result. \square

Fix for the whole paper a number $\delta > 0$ so small that

$$0 < \delta < \frac{\pi}{4}$$

and such that $(-\Delta_{k_2} - \lambda\varepsilon_0)^{-1}$ exists for all k_2 in the following set:

$$Z_0 := \{z \in \mathbb{C} : |\operatorname{Im} z| < \delta\}.$$

Such choice of δ is possible by remark 2.1 and (2). Define moreover

$$Z := \{z = \xi_2 + i\tau_2 : \xi_2 \in (\pi - \delta, \pi + \delta), \tau_2 \in \mathbb{R}\}.$$

The reader should keep in mind that we regard Z_0, Z as sets in the complex k_2 -plane.

5. CONTINUATION OF RESOLVENT OPERATORS.

Theorem 5.1. *There exists a number $\tau_1 \in 2\pi\mathbb{N}$ such that for all $k_2 \in Z$, $T(k_1, k_2)$ exists for all $k_1 \in [-\pi, \pi] \pm i\tau_1$.*

Proof. From theorem 8.4 (in particular estimate (14)) we get

$$\| -\Delta_{(\xi_1 + i\tau_1, k_2)}^{-1} \| \leq \left(\min_{m_2 \in \mathbb{Z}} |(m_2 + \operatorname{Re} k_2)^2 - \tau_1^2| \right)^{-1}.$$

for $\xi_1 \in [-\pi, \pi]$, $\tau_1 \in \mathbb{R}$. Since $\operatorname{Re} k_2 \in [\pi - \delta, \pi + \delta]$, by lemma 8.4 we may choose $\tau_1 \in 2\pi\mathbb{N}$ so large that for all $\xi_1 \in [-\pi, \pi]$

$$\| -\Delta_{(\xi_1 + i\tau_1, k_2)}^{-1} \| \leq \frac{1}{2\lambda \|\varepsilon_0\|_\infty} \quad (k_2 \in Z)$$

holds. The standard Neumann series argument then shows that

$$(-\Delta_{(\xi_1 + i\tau_1, z_2)} - \lambda\varepsilon_0)^{-1}$$

exists for $\xi_1 \in [-\pi, \pi]$ and z_2 in a small neighborhood of k_2 . \square

5.1. Construction of the analytic continuation. We will now describe the analytic continuation of the operator

$$G(k_2)r = \varepsilon_1(-\Delta_{k_2} - \lambda\varepsilon_0)^{-1}\tilde{r}$$

to values k_2 with large imaginary part, along a certain path Γ in the complex k_2 -plane.

In the first step, choose a number

$$\tau_1 \in 2\pi\mathbb{N},$$

which will be fixed for the rest of the whole paper, with the properties from theorem 5.1, i.e., for all $k_2 \in Z$, $T(k_1, k_2)$ exists for k_1 on the complex lines $[-\pi, \pi] \pm i\tau_1$. So by the analytic Fredholm theorem, $T(k_1, k_2)$ exists for all except for a discrete set of k_1 's in the complex plane, being the poles of $T(\cdot, k_2)$. By the poles of $T(\cdot, k_2)$, we mean the poles of the meromorphic function

$$k_1 \mapsto T(k_1, k_2).$$

In the following we will consider the position of the poles of $T(\cdot, k_2)$ lying inside

$$D = \{z \in \mathbb{C} : |\operatorname{Im} z| < \tau_1, \operatorname{Re} z \in [-\pi, \pi]\}.$$

for fixed $k_2 \in Z$. Notice that if k is a pole of $T(\cdot, k_2)$, then $k + m$, $m \in 2\pi\mathbb{Z}$ is also a pole. This observation is important, since we will track the position of the poles in the complex plane modulo $2\pi\mathbb{Z}$ in the real part. We have already reflected this fact in the definition of D above.

Define for $r \in L^2(\Omega)$

$$H(k_1, k_2)r := e^{ik_1 x_1} T(k_1, k_2)[e^{-ik_1 \cdot} r]$$

($e^{-ik_1 \cdot}$ means the function $(y_1, y_2) \mapsto e^{-ik_1 y_1}$) and note that

$$(8) \quad H(k_1 + 2\pi m, k_2) = H(k_1, k_2) \quad (m \in \mathbb{Z}).$$

holds, whenever k_1 is not a pole of $T(\cdot, k_2)$.

Whenever we speak of the number of poles of $T(\cdot, k_2)$, we do not take into account their orders, e.g. the number of poles of the meromorphic function $z \mapsto z^{-2} + (z-1)^{-3}$ is two.

Lemma 5.1. *There exist a continuous path*

$$\Gamma : [0, \infty) \rightarrow Z$$

satisfying

- (i) $\Gamma(0) \in \mathbb{R}$,
- (ii) $t \mapsto \operatorname{Im} \Gamma(t)$ is nondecreasing,
- (iii) $\operatorname{Im} \Gamma(t) \rightarrow +\infty$ for $t \rightarrow \infty$,

with the property that there exists a neighborhood

$$\mathcal{N}(\Gamma) := \mathcal{N}(\Gamma([0, \infty)))$$

of the path Γ and a $N \in \mathbb{N}$ such that the number of poles of $T(\cdot, k_2)$ in D is equal to N for all $k_2 \in \mathcal{N}(\Gamma)$.

Proof. According to theorem 8.3, the number of poles of $T(\cdot, k_2)$ is equal to some number $N \in \mathbb{N}$, as k_2 varies in Z , except in case when $k_2 \in \mathcal{E}$. Here \mathcal{E} , is a certain discrete set of exceptional points not accumulating anywhere in \overline{Z} . So it is possible to choose a continuous path Γ with the desired properties (see figure 2 for an illustration). \square

Definition 5.2. *For $k_2 \in \mathcal{N}(\Gamma)$ let*

$$q_j^+(k_2) \quad (j = 1, \dots, N^+)$$

denote the poles of $T(\cdot, k_2)$ in D with the property that $\operatorname{Im} q_j^+(\Gamma(0)) > 0$, i.e. those poles which initially lie in the upper half-plane. For any $k_2 \in \mathcal{N}(\Gamma)$ define

$$(9) \quad A(k_2)r := \int_{[-\pi, \pi] + i\tau_1} H(k_1, k_2)r \, dk_1 + 2\pi i \sum_{j=1}^{N^+} \operatorname{res}(H(\cdot, k_2)r, q_j^+(k_2))$$

for all $r \in L^2(\Omega)$. The formula (9) is understood as a function in $L^2(\Omega)$.

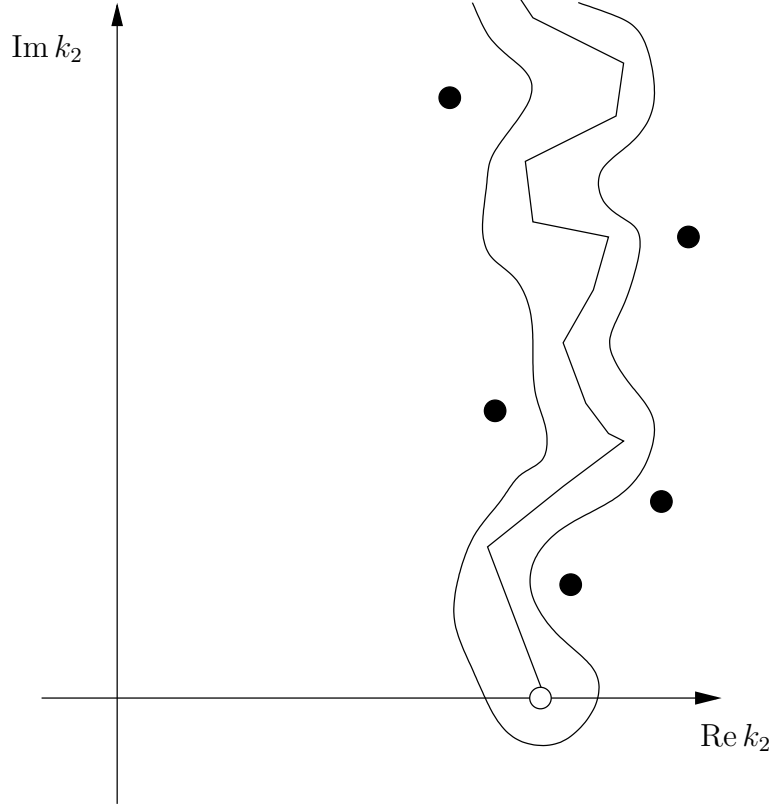
$$\operatorname{res}(H(\cdot, k_2)r, q_j^+(k_2))$$

denotes the residual of the meromorphic $L^2(\Omega)$ -valued function $k_1 \mapsto H(k_1, k_2)r$ at the pole $q_j^+(k_2)$.

In general, the $q_j^+(k_2)$ are algebraic functions of k_2 , i.e. they may behave like complex roots in the vicinity of points of the exceptional set \mathcal{E} (compare theorem 8.3 and the discussion following it). On $\mathcal{N}(\Gamma)$ however, they are analytic.

Lemma 5.2. *$A(k_2) : L^2(\Omega) \rightarrow L^2(\Omega)$ is a compact operator for each $k_2 \in \mathcal{N}(\Gamma)$; moreover, $k_2 \mapsto A(k_2)$ is an analytic operator-valued function.*

FIGURE 2. The path Γ and the neighborhood $\mathcal{N}(\Gamma)$ in the k_2 -plane. The solid circles indicate points in the exceptional set \mathcal{E} .



Proof. The compactness is implied by the standard estimate

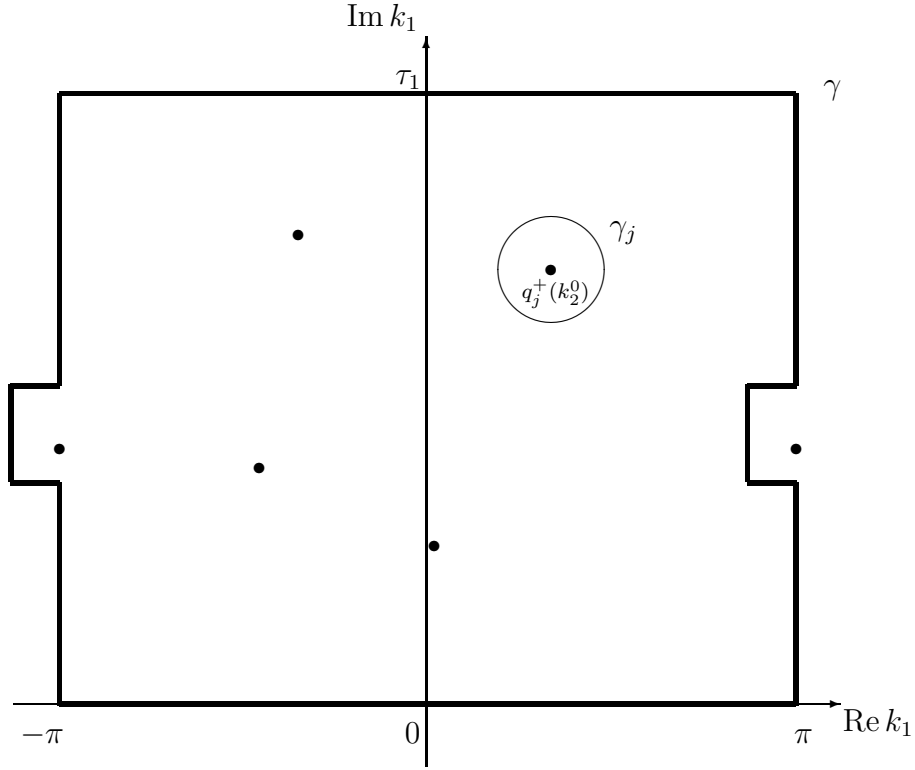
$$\|\nabla T(k_1, k_2)f\|_{L^2(\Omega)} \leq C(k_1, k_2) [\|f\|_{L^2(\Omega)} + \|T(k_1, k_2)f\|_{L^2(\Omega)}].$$

The integral over $[-\pi, \pi] + i\tau_1$ in (9) depends analytically on k_2 , since $H(k_1, k_2)$ exists for all $k_2 \in \mathcal{N}(\Gamma)$ and depends analytically on k_2 . In order to prove the analyticity of the sum in (9), fix a $k_2^0 \in \mathcal{N}(\Gamma)$ and choose a system of small circles γ_j in the complex k_1 -plane, each of the γ_j enclosing one of the $q_j^+(k_2^0)$ and no other poles. Since the number of poles stays constant away from the set \mathcal{E} from theorem 8.3, each of the γ_j encloses exactly the pole $q_j^+(k_2)$ for k_2 in a small neighborhood of k_2^0 . Hence the sum in (9) may be written as

$$\sum_{j=1}^{N^+} \oint_{\gamma_j} H(k_1, k_2)r \, dk_1$$

for all k_2 close to k_2^0 and we see that it is obviously analytic in k_2 . \square

The next lemma shows that indeed $A(k_2)r$ is an analytic continuation of $(-\Delta_{k_2} - \lambda\varepsilon_0)^{-1}\tilde{r}$.

FIGURE 3. The contours γ and γ_j used in the proof of lemma 5.2 and 5.3

Lemma 5.3. *In $Z_0 \cap \mathcal{N}(\Gamma)$,*

$$(10) \quad A(k_2)r = (-\Delta_{k_2} - \lambda\varepsilon_0)^{-1}\tilde{r}$$

for all $r \in L^2(\Omega)$ ($\tilde{r} = r$ on Ω and zero on $S \setminus \Omega$).

Proof. Choose a contour $\gamma = \gamma(k_2)$ in the complex plane as indicated in figure 3, where the lateral parts of γ avoid poles $q_j^+(k_2)$ with real part equal to $-\pi$ or π . The lateral part to the right has the same shape as the left part, but it is shifted by 2π in positive real direction.

None of the $q_j^+(k_2)$ lies on the real axis for k_2 close to $\Gamma(0)$, since $\Gamma(0) \in Z_0$. Using the residue theorem, we get

$$(11) \quad \int_{\gamma} H(k_1, k_2)r \, dk_1 = 2\pi i \sum_{j=1}^{N^+} \text{res}(H(\cdot, k_2)r, q_j^+(k_2)),$$

since the $q_j^+(k_2)$ are exactly the poles in the upper half-plane for k_2 close to $\Gamma(0)$. Since the contributions from the lateral parts of the contour cancel due to the

periodicity (8), the left-hand side of (11) is just

$$\begin{aligned} \int_{[-\pi, \pi]} H(k_1, k_2) r \, dk_1 - \int_{[-\pi, \pi] + i\tau_1} H(k_1, k_2) r \, dk_1 = \\ (-\Delta_{k_2} - \lambda \varepsilon_0)^{-1} \tilde{r} - \int_{[-\pi, \pi] + i\tau_1} H(k_1, k_2) r \, dk_1, \end{aligned}$$

where we also used (2). This proves the lemma. \square

Remark 5.3. For $k_2 \in \mathcal{N}(\Gamma)$ outside Z_0 , the relation (10) does not hold in general, but this will not matter in the following.

6. ASYMPTOTIC BEHAVIOR OF $A(k_2)$ AS $\text{Im } k_2 \rightarrow \infty$

6.1. Asymptotic localization of the poles of $T(\cdot, k_2)$.

Definition 6.1. For $m_2 \in 2\pi\mathbb{Z}$, $-\tau_1 + 2\pi \leq m_2 \leq \tau_1$ we define the following contours in the complex k_1 -plane:

$$\Gamma_{m_2}^{\pm} := \left\{ k_1 \in \mathbb{C} : \max \left\{ \left| \frac{\text{Re } k_1 \mp \frac{\pi}{2}}{2\delta} \right|, \left| \frac{\text{Im } k_1 - m_2 + \pi}{\pi} \right| \right\} = 1 \right\}.$$

The most important features of these contours are $\Gamma_{m_2}^{\pm} \subseteq \text{Lines}$ (see definition 8.5 and figure 4) and the following

Lemma 6.1. There exists a number $M = M(\delta, \tau_1, \lambda) > 0$ such that for all $k_2 \in \mathcal{N}(\Gamma)$ with $\text{Im } k_2 = \frac{\pi}{2} + \ell$, $\ell \in 2\pi\mathbb{N}$, $\ell > M$, each of the $q_j^+(k_2)$ is enclosed by one of the contours $\Gamma_{m_2}^{\pm}$. Moreover, each contour encloses exactly one of the $q_j^+(k_2)$ and no other pole of $T(\cdot, k_2)$.

Proof. As in definition 8.2 we define the operator $W_{\mu}(k_2)$. Lemma 8.1 shows that k_1 is a pole of $(-\Delta_{(k_1, k_2)} - \mu \varepsilon_0)^{-1}$ if and only if k_1 is an eigenvalue of $W_{\mu}(k_2)$. From lemma 8.2 we see that, if $\text{Im } k_2 = \frac{\pi}{2} + \ell$, each of the contours $\Gamma_{m_2}^{\pm}$ encloses exactly one pole of $-\Delta_{(\cdot, k_2)}^{-1}$, i.e. each $\Gamma_{m_2}^{\pm}$ encloses exactly one eigenvalue of $W_0(k_2)$. Moreover, from lemma 8.2 follows that the dimension of the corresponding eigenspace is one. Then, since $\Gamma_{m_2}^{\pm} \subseteq \text{Lines}$, by (17) there exists a $M = M(\delta, \tau_1, \lambda)$ such that for $\ell > M$ the norm of $\lambda(-\Delta_{(k_1, k_2)})^{-1} \varepsilon_0$ is less than 1 on all the contours $\Gamma_{m_2}^{\pm}$. By a Neumann series argument, $(-\Delta_{(k_1, k_2)} - \mu \varepsilon_0)^{-1}$ exists for all $\mu \in [0, \lambda]$ on the contours and hence the dimension of the range of the projection

$$\frac{1}{2\pi i} \oint_{\Gamma_{m_2}^{\pm}} (W_{\mu}(k_2) - k)^{-1} dk$$

does not change when μ varies in $[0, \lambda]$. But this implies that each of the $\Gamma_{m_2}^{\pm}$ encloses exactly one pole of $T(\cdot, k_2)$. In particular, each of the $q_j^+(k_2)$ must be enclosed by exactly one of the contours. \square

6.2. Estimate for $\text{Im } k_2$ large.

Theorem 6.2. *There exist constants $C = C(\delta, \tau_1, \lambda) > 0$, $M = M(\delta, \tau_1, \lambda) > 0$ such that for $k_2 \in \mathcal{N}(\Gamma)$ of the form $k_2 = \text{Re } k_2 + i(\frac{\pi}{2} + \ell)$ with $\ell \in 2\pi\mathbb{N}$, $\ell > M$,*

$$\|A(k_2)\| \leq C\ell^{-1}.$$

Proof. Let $k_1 \in [-\pi, \pi] + i\tau_1$ or $k_1 \in \Gamma_{m_2}^\pm$. Since $([-\pi, \pi] + i\tau_1) \cup \Gamma_{m_2}^\pm \subseteq \text{Lines}$, by corollary 8.1 there exists a $C = C(\delta, \tau_1, \lambda)$ and a $M = M(\delta, \tau_1, \lambda)$ such that

$$\|T(k_1, k_2)\| \leq C/\ell$$

for all $\ell > M$. This gives

$$(12) \quad \|H(k_1, k_2)r\|_{L^2(\Omega)} \leq \|e^{-ik_1 \cdot} T(k_1, k_2)(e^{-ik_1 \cdot} r)\|_{L^2(\Omega)} \leq C\ell^{-1} \|r\|_{L^2(\Omega)}$$

with another constant $C = C(\delta, \tau_1, \lambda)$ independent of $\ell > M$ (since k_1 is from a bounded region in the complex plane). It suffices to estimate the integral and the sum in (9) separately. Using (12), the L^2 -norm of the integral over $[-\pi, \pi] + i\tau_1$ is easily estimated by $C\ell^{-1} \|r\|_{L^2(\Omega)}$ with another constant $C = C(\delta, \tau_1, \lambda)$ independent of $\ell > M$. On the other hand, each pole $q_j^+(k_2)$ lies in exactly one of the contours $\Gamma_{m_2}^\pm$ and hence the residue $\text{res}(H(\cdot, k_2)r, q_j^+(k_2))$ can be expressed as

$$\frac{1}{2\pi i} \oint_{\Gamma_{m_2}^\pm} H(k, k_2) dk$$

which together with (12), immediately implies the estimate

$$\left\| 2\pi i \sum_{j=1}^{N^+} \text{res}(H(\cdot, k_2)r, q_j^+(k_2)) \right\|_{L^2(\Omega)} \leq C\ell^{-1} \|r\|_{L^2(\Omega)}$$

with another constant $C = C(\delta, \tau_1, \lambda)$ independent of $\ell > M$. \square

7. PROOF OF THE MAIN THEOREM

Proof of theorem 3.1. Consider the analytic Fredholm equation

$$(13) \quad v + \lambda \varepsilon_1 A(k_2)v = 0 \quad \text{on } \Omega$$

for the unknown function $v \in L^2(\Omega)$, and with $k_2 \in \mathcal{N}(\Gamma)$. By theorem 6.2, (13) has only the trivial solution if $k_2 \in \mathcal{N}(\Gamma)$ with $\text{Im } k_2 = \frac{\pi}{2} + \ell$ and $\ell \in 2\pi\mathbb{N}$ is sufficiently large. By the analytic Fredholm theorem, the set where (13) has a nontrivial solution is discrete in $\mathcal{N}(\Gamma)$. From lemma 5.3, we see that the problem (13) coincides with (7) for all $k_2 \in Z_0 \cap \mathcal{N}(\Gamma)$, hence (7) has a nontrivial solution only for a discrete set of $k_2 \in Z_0$.

Now if (1) has a nontrivial solution $u \in H^2(\mathbb{R}^2)$, then (7) has a nontrivial solution for all real k_2 , yielding a contradiction. \square

8. APPENDIX

8.1. Poles of resolvent operators.

Theorem 8.1. *The poles of $T(k_1, k_2)$ in k_1 contained in the set*

$$D = \{z \in \mathbb{C} : |\operatorname{Im} z| < \tau_1, \operatorname{Re} z \in [-\pi, \pi)\}$$

are locally in k_2 the zeros of an analytic function depending on k_1, k_2 . More precisely, for each $k_2 \in Z$, there exists a neighborhood $\mathcal{N} = \mathcal{N}(k_2)$ and an analytic function $F : \mathbb{C} \times \mathcal{N} \rightarrow \mathbb{C}$ such that for $z_2 \in \mathcal{N}$, $z \in D$ is a pole of $T(z, z_2)$ if and only if $F(z, z_2) = 0$.

Before proving the theorem, we need to define the following operator and state some of its easily proved properties.

Definition 8.2. *Let $W_\lambda(k_2) : D(W_\lambda(k_2)) = H_{\text{per}}^2(\Omega) \times H_{\text{per}}^1(\Omega) \rightarrow H_{\text{per}}^1(\Omega) \times L^2(\Omega)$ be defined by*

$$W_\lambda(k_2)(u, v) := (v, \Delta_{(0, k_2)}u + 2i\partial_1 v + \lambda\varepsilon_0 u).$$

We regard $W_\lambda(k_2)$ as an unbounded operator in the Hilbert space $H_{\text{per}}^1(\Omega) \times L^2(\Omega)$.

Lemma 8.1. *$(-\Delta_{(k_1, k_2)} - \lambda\varepsilon_0)^{-1}$ exists if and only if $(W_\lambda(k_2) - k_1)^{-1}$ exists. As a consequence, k_1 is a pole of $T(\cdot, k_2)$ if and only if k_1 is an eigenvalue of $W_\lambda(k_2)$. Moreover,*

$$\ker(W_\lambda(k_2) - k_1) = \{(u, k_1 u) : u \in \ker(-\Delta_{(k_1, k_2)} - \lambda\varepsilon_0)\}.$$

Proof of theorem 8.1. In order to study the poles of $T(\cdot, z_2)$, it is sufficient by lemma 8.1 to study the eigenvalues of $W_\lambda(k_2)$ contained in D .

Now fix $k_2 \in Z$ and choose an $\theta > 0$ such that the boundary of

$$\tilde{D} := \{z \in \mathbb{C} : \operatorname{Re} z \in (-\pi - \theta, \pi + \theta), |\operatorname{Im} z| < \tau_1\}$$

is free of eigenvalues of $W_\lambda(k_2)$; then the Riesz projection

$$P(z_2) = \frac{1}{2\pi i} \int_{\partial \tilde{D}} (W_\lambda(z_2) - z)^{-1} dz$$

is analytic for z_2 in a small neighborhood \mathcal{N} of k_2 . For all $z_2 \in \mathcal{N}$, $z \in D$ is an eigenvalue of $W_\lambda(z_2)$ if and only if $W_\lambda(z_2)$ has z as an eigenvalue, considered as an operator on the invariant finite-dimensional space $\operatorname{ran}(P(z_2))$. Standard arguments from perturbation theory (see [7]) assure us that there exists an isomorphism (analytic in z_2) $U(z_2) : \operatorname{ran}(P(k_2)) \rightarrow \operatorname{ran}(P(z_2))$ for $z_2 \in \mathcal{N}$. Hence,

$$F(z_1, z_2) := \det[U^{-1}(z_2)(W_\lambda(z_2) - z_1)U(z_2)] \quad ((z_1, z_2) \in \mathbb{C} \times \mathcal{N})$$

is an analytic function having the required properties. \square

Theorem 8.3. *There exists a set $\mathcal{E} \subset \mathbb{C}$ which does not accumulate in \overline{Z} and a natural number $N \in \mathbb{N}$, such that the number of poles of $T(\cdot, k_2)$ contained in D is equal to N for all $k_2 \in Z \setminus \mathcal{E}$.*

This follows from theorem 8.1 and well-known facts from complex analysis (follow the references in [7], chapter 2). Note that despite the fact that theorem 8.1 only gives a local (in k_2) description of the poles of $T(\cdot, k_2)$, simple compactness arguments give the global constancy of the number of poles in $Z \setminus \mathcal{E}$. An alternative suggestive reason for the constancy is the following: because of theorem 5.1, the poles cannot cross the upper and lower boundary of D and thus are confined inside/outside D (recall that we track the position of the poles modulo 2π in real direction). So the only way the number of poles inside D can change as k_2 varies, is by collision, which happens exactly if $k_2 \in \mathcal{E}$.

8.2. Estimates for the symbol of $-\Delta_{\mathbf{k}}$ for complex \mathbf{k} . Recall that $s(\mathbf{m}, \mathbf{k}) = (\mathbf{m} + \mathbf{k})^2$ is the symbol of the operator $-\Delta_{\mathbf{k}}$ in the Fourier series representation. The following estimates for the symbol are completely elementary, yet they play a crucial role in our whole development.

Theorem 8.4. *For $\boldsymbol{\xi} = (\xi_1, \xi_2)$, $\boldsymbol{\eta} = (\eta_1, \eta_2) \in \mathbb{R}^2$, $\mathbf{m} = (m_1, m_2) \in 2\pi\mathbb{Z}^2$ the following estimates hold:*

$$(14) \quad |s(\mathbf{m}, \boldsymbol{\xi} + i\boldsymbol{\eta})|^2 \geq [(m_2 + \xi_2)^2 - \eta_1^2]^2 + [(m_1 + \xi_1)^2 - \eta_2^2]^2$$

$$(15) \quad |s(\mathbf{m}, \boldsymbol{\xi} + i\boldsymbol{\eta})|^2 \geq 2[(m_1 + \xi_1)\eta_1 + (m_2 + \xi_2)\eta_2]^2$$

Proof. First note

$$|s(\mathbf{m}, \boldsymbol{\xi} + i\boldsymbol{\eta})|^2 = [(\mathbf{m} + \boldsymbol{\xi})^2 - \boldsymbol{\eta}^2]^2 + 4[(\mathbf{m} + \boldsymbol{\xi}) \cdot \boldsymbol{\eta}]^2.$$

Now note the following identity (χ_1, χ_2 real numbers)

$$\begin{aligned} & (\chi_2^2 - \eta_1^2 + \chi_1^2 - \eta_2^2)^2 + 4(\chi_1\eta_1 + \chi_2\eta_2)^2 \\ &= (\chi_2^2 - \eta_1^2)^2 + (\chi_1^2 - \eta_2^2)^2 + 2(\chi_1\eta_1 + \chi_2\eta_2)^2 + 2(\chi_2\chi_1 + \eta_1\eta_2)^2. \end{aligned}$$

Apply this with $\chi_i = m_i + \xi_i$, $i = 1, 2$, to obtain

$$(16) \quad \begin{aligned} |s(\mathbf{m}, \boldsymbol{\xi} + i\boldsymbol{\eta})|^2 &= [(m_2 + \xi_2)^2 - \eta_1^2]^2 + [(m_1 + \xi_1)^2 - \eta_2^2]^2 \\ &\quad + 2[(m_1 + \xi_1)\eta_1 + (m_2 + \xi_2)\eta_2]^2 \\ &\quad + 2[(m_2 + \xi_2)(m_1 + \xi_1) + \eta_1\eta_2]^2. \end{aligned}$$

From this both estimates follow. \square

Fix a $k_2 = \xi_2 + i\eta_2$ with $\xi_2 \in [\pi - \delta, \pi + \delta]$ and $\eta_2 = \frac{\pi}{2} + \ell$, $\ell \in 2\pi\mathbb{N}_0$. Define $\mathcal{J}_+, \mathcal{J}_- : 2\pi\mathbb{Z}^2 \rightarrow \mathbb{C}$ by

$$\begin{aligned} \mathcal{J}_{\pm}(m_1, m_2) &= \pm\left(\frac{\pi}{2} + \ell\right) - m_1 \mp i|m_2 + \xi_2| \quad \text{if } m_2 \geq 0 \\ \mathcal{J}_{\pm}(m_1, m_2) &= \pm\left(\frac{\pi}{2} + \ell\right) - m_1 \pm i|m_2 + \xi_2| \quad \text{if } m_2 < 0. \end{aligned}$$

Note that \mathcal{J}_+ and \mathcal{J}_- are one to one and

$$\mathcal{J}_+(2\pi\mathbb{Z}^2) \cap \mathcal{J}_-(2\pi\mathbb{Z}^2) = \emptyset.$$

In the next lemma, the poles of $(-\Delta_{(\cdot, \xi_2 + i\eta_2)})^{-1}$ are determined. The proof is a simple computation using the above defined \mathcal{J}_{\pm} and (16).

Lemma 8.2. *Let $k_2 = \xi_2 + i\eta_2$ with $\xi_2 \in [\pi - \delta, \pi + \delta]$ and $\eta_2 = \frac{\pi}{2} + \ell$, $\ell \in 2\pi\mathbb{N}_0$ be fixed. Then*

- (i) $s(\mathbf{m}, (k_1, \xi_2 + i\eta_2)) = 0$ if and only if $k_1 = \mathcal{J}_+(\mathbf{m})$ or $k_1 = \mathcal{J}_-(\mathbf{m})$.
- (ii) $\ker(-\Delta_{(k_1, \xi_2 + i\eta_2)}) = \text{span}\{e^{i\mathbf{m}\cdot\mathbf{x}}\}$, where $\mathbf{m} \in 2\pi\mathbb{Z}^2$ is uniquely determined by the condition $k_1 = \mathcal{J}_+(\mathbf{m})$ or $k_1 = \mathcal{J}_-(\mathbf{m})$. If there is no $\mathbf{m} \in 2\pi\mathbb{Z}^2$ satisfying $k_1 = \mathcal{J}_+(\mathbf{m})$ or $k_1 = \mathcal{J}_-(\mathbf{m})$, then $-\Delta_{(k_1, \xi_2 + i\eta_2)}$ is invertible.

As a consequence, for any pole k_1 of $-\Delta_{(\cdot, \xi_2 + i\eta_2)}^{-1}$ we have either $k_1 = \mathcal{J}_+(\mathbf{m})$ or $k_1 = \mathcal{J}_-(\mathbf{m})$ with a uniquely determined $\mathbf{m} \in 2\pi\mathbb{Z}^2$. Moreover, if k_1 is a pole of $-\Delta_{(\cdot, \xi_2 + i\eta_2)}^{-1}$, then $k_1 + m$, $m \in 2\pi\mathbb{Z}$ is also a pole.

Definition 8.5. We define the set $\text{Lines} \subseteq \mathbb{C}$ consisting of four vertical and a finite family of horizontal lines in the complex k_1 -plane (see figure 4) by

$$\text{Lines} := \left((\pm \frac{\pi}{2} \pm 2\delta) + i\mathbb{R} \right) \cup \left(\bigcup_{\nu \in 2\pi\mathbb{Z}, |\nu| \leq \tau_1} [-\pi, \pi] + i\nu \right).$$

Lemma 8.3. *There exists a $C = C(\delta, \tau_1) > 0$ and a $M = M(\delta, \tau_1) > 0$ such that for all $\ell \in 2\pi\mathbb{N}$, $\ell > M$, all $k_1 \in \text{Lines}$, and all $\xi_2 \in [\pi - \delta, \pi + \delta]$ the following estimate for the symbol of $-\Delta_{\mathbf{k}}$ holds:*

$$\left| s(\mathbf{m}, (k_1, \xi_2 + i(\frac{\pi}{2} + \ell))) \right| \geq C\ell \quad (\mathbf{m} \in 2\pi\mathbb{Z}^2).$$

As a consequence,

$$(17) \quad \left\| (-\Delta_{(k_1, \xi_2 + i(\frac{\pi}{2} + \ell))})^{-1} \right\| \leq C\ell^{-1}$$

for all $k_1 \in \text{Lines}$, $\ell > M$, $\xi_2 \in [\pi - \delta, \pi + \delta]$.

Proof lemma 8.3. In total we have to consider four cases:

1. Vertical lines: $k_1 = (\pm \frac{\pi}{2} \pm 2\delta) + i\nu$ ($\nu \in \mathbb{R}$)
 - Case 1.1 $\mathbf{m} = (\pm \ell, m_2)$
 - Case 1.2 $\mathbf{m} = (m_1, m_2)$ with $m_1 \neq \pm \ell$
2. Horizontal lines: $k_1 = \mu + i\nu$ ($\mu \in [-\pi, \pi]$, $\nu \in 2\pi\mathbb{Z}$, $|\nu| \leq \tau_1$)
 - Case 2.1 $\mathbf{m} = (\pm \ell, m_2)$
 - Case 2.2 $\mathbf{m} = (m_1, m_2)$ with $m_1 \neq \pm \ell$

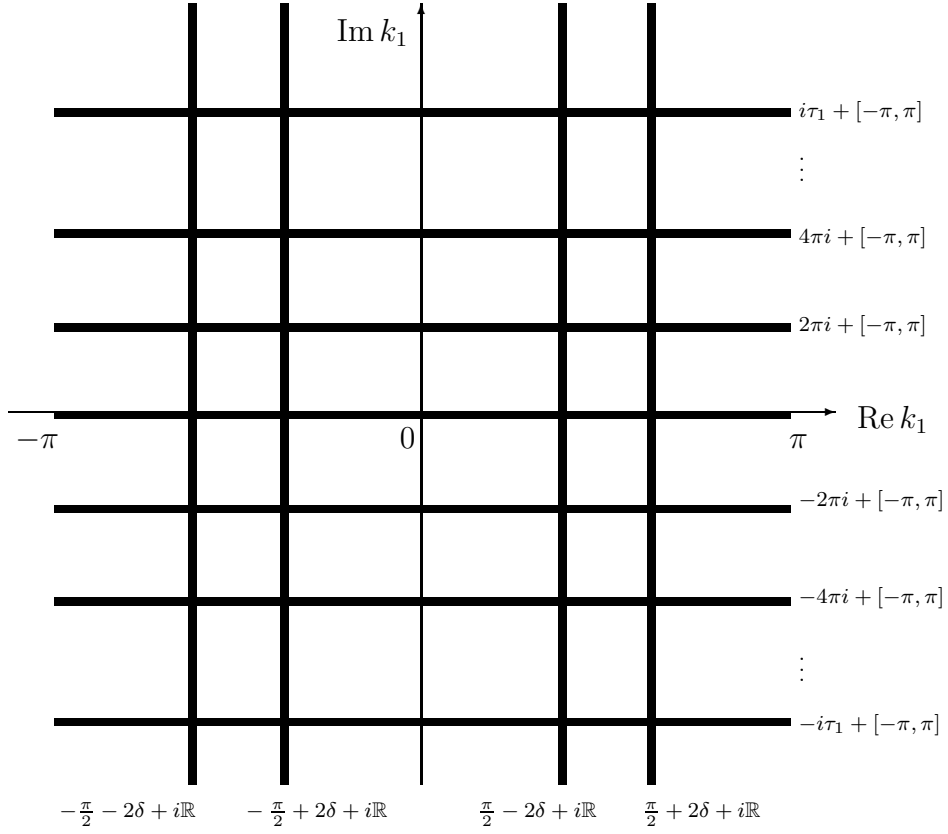
For the case 1.1, we use the estimate (14) to obtain

$$\begin{aligned} \left| s(\mathbf{m}, (k_1, \xi_2 + i(\frac{\pi}{2} + \ell))) \right|^2 &\geq [(\pm \ell + (\pm \frac{\pi}{2} \pm 2\delta))^2 - (\frac{\pi}{2} + \ell)^2]^2 \\ &= [\pm 2\ell(\pm \frac{\pi}{2} \mp \frac{\pi}{2} \pm 2\delta) + (\pm \frac{\pi}{2} \pm 2\delta)^2 - \frac{\pi^2}{4}]^2 \\ &\geq C(\delta)\ell^2 \end{aligned}$$

for sufficiently large $\ell \in 2\pi\mathbb{N}$, since $(\pm \frac{\pi}{2} \mp \frac{\pi}{2} \pm 2\delta) \neq 0$ by the choice $0 < 2\delta < \frac{\pi}{2}$. To treat the cases 1.2 and 2.2 we consider the intervals

$$I_m := (m + [-\pi, \pi])^2 = \{(m + \eta)^2 : \eta \in [-\pi, \pi]\},$$

FIGURE 4. Illustration of the set Lines in the complex plane.



where $m \in 2\pi\mathbb{Z}$. Then $I_m = I_{-m}$ and $\max I_{|m|} = \min I_{|m|+2\pi}$. So the intervals $I_{|m|}$ and $I_{|m|+2\pi}$ are adjacent and the union of all I_m is $[0, \infty)$. Since $(\frac{\pi}{2} + \ell)^2 \in I_\ell$ and $m_1 \neq \pm\ell$ we have for sufficiently large ℓ

$$\begin{aligned}
 \text{dist}\left(\left(\frac{\pi}{2} + \ell\right)^2, I_{m_1}\right) &\geq \min \left\{ \left(\frac{\pi}{2} + \ell\right)^2 - (\ell - \pi)^2, (\ell + \pi)^2 - \left(\frac{\pi}{2} + \ell\right)^2 \right\} \\
 (18) \qquad \qquad \qquad &\geq (\pi\ell + \frac{3}{4}\pi^2) \geq C\ell
 \end{aligned}$$

with some constant $C > 0$. Using (14) we obtain

$$\left| s(\mathbf{m}, (\mu + i\nu, \xi_2 + i\left(\frac{\pi}{2} + \ell\right))) \right|^2 \geq [(m_1 + \mu)^2 - \left(\frac{\pi}{2} + \ell\right)^2]^2 \geq C^2 \ell^2$$

by (18) since $(m_1 + \mu)^2 \in I_{m_1}$. This proves the desired estimate in the case 2.2. In the case 1.2, the proof is the same since again by estimate (14)

$$\left| s(\mathbf{m}, ((\pm\frac{\pi}{2} \pm 2\delta) + i\nu, \xi_2 + i\left(\frac{\pi}{2} + \ell\right))) \right|^2 \geq [(m_1 + (\pm\frac{\pi}{2} \pm 2\delta))^2 - \left(\frac{\pi}{2} + \ell\right)^2]^2$$

and $(m_1 + (\pm\frac{\pi}{2} \pm 2\delta))^2 \in I_{m_1}$.

For the case 2.1 we use the estimate (15).

$$\begin{aligned} \left| s(\mathbf{m}, (\mu + i\nu, \xi_2 + i\left(\frac{\pi}{2} + \ell\right))) \right|^2 &\geq 2[(\pm\ell + \mu)\nu + (m_2 + \xi_2)\left(\frac{\pi}{2} + \ell\right)]^2 \\ &= 2\left(\frac{\pi}{2} + \ell\right)^2 \left[m_2 + \frac{\pm\ell + \mu}{\left(\frac{\pi}{2} + \ell\right)}\nu + \xi_2 \right]^2. \end{aligned}$$

Since $m_2 + \frac{\pm\ell + \mu}{\left(\frac{\pi}{2} + \ell\right)}\nu$ converges (uniformly in $\mu \in [-\pi, \pi]$ and $\nu \in 2\pi\mathbb{Z}$, $|\nu| \leq \tau_1$) to some $\widetilde{m}_\pm \in 2\pi\mathbb{Z}$ as $\ell \rightarrow \infty$, and since $\xi_2 \in [\pi - \delta, \pi + \delta]$ with $0 < \delta < \frac{\pi}{4}$, there exists a constant $C(\delta, \tau_1) > 0$ such that for sufficiently large ℓ

$$\left[m_2 + \frac{\pm\ell + \mu}{\left(\frac{\pi}{2} + \ell\right)}\nu + \xi_2 \right]^2 \geq C(\delta, \tau_1) > 0.$$

Then, for sufficiently large ℓ

$$\left| s(\mathbf{m}, (\mu + i\nu, \xi_2 + i\left(\frac{\pi}{2} + \ell\right))) \right|^2 \geq C(\delta, \tau_1)\ell^2$$

holds, with another constant $C(\delta, \tau_1) > 0$. \square

Using the same Neumann series argument as in the proof of theorem 5.1 we obtain the following

Corollary 8.1. *There exists a $C = C(\delta, \tau_1, \lambda) > 0$ and a $M = M(\delta, \tau_1, \lambda) > 0$ such that for all $\ell \in 2\pi\mathbb{N}$, $\ell > M$, all $k_1 \in \text{Lines}$, and all $\xi_2 \in [\pi - \delta, \pi + \delta]$ the following estimate holds:*

$$\left\| T(k_1, \xi_2 + i\left(\frac{\pi}{2} + \ell\right)) \right\| \leq C\ell^{-1}.$$

Lemma 8.4. *Let $0 < \delta < \frac{\pi}{4}$, $\xi_2 \in [\pi - \delta, \pi + \delta]$. Then for $m \in 2\pi\mathbb{N}_0$*

$$\min_{m_2 \in 2\pi\mathbb{Z}} |(m_2 + \xi_2)^2 - (m + 2\pi)^2| \geq (2m + 3\pi + \delta)(\pi - \delta).$$

Proof. First notice that

$$\bigcup_{m_2 \in 2\pi\mathbb{Z}} (m_2 + [\pi - \delta, \pi + \delta])^2 = \bigcup_{m \in 2\pi\mathbb{N}_0} I_m^+ \cup \bigcup_{m \in 2\pi\mathbb{N}} I_m^-$$

where

$$I_m^\pm = (\pm m + [\pi - \delta, \pi + \delta])^2 = [(\pm m + \pi \mp \delta)^2, (\pm m + \pi \pm \delta)^2].$$

Moreover, $I_{m+2\pi}^- = I_m^+$ for $m \in 2\pi\mathbb{N}_0$, i.e.

$$\bigcup_{m_2 \in 2\pi\mathbb{Z}} (m_2 + [\pi - \delta, \pi + \delta])^2 = \bigcup_{m \in 2\pi\mathbb{N}_0} I_m^+.$$

The intervals I_m^+ and $I_{m+2\pi}^+$ are disjoint and $(m + 2\pi)^2$ lies in the gap between them. Moreover,

$$\begin{aligned} \min I_{m+2\pi}^+ - (m + 2\pi)^2 &= (2m + 5\pi - \delta)(\pi - \delta) \\ (m + 2\pi)^2 - \max I_m^+ &= (2m + 3\pi + \delta)(\pi - \delta), \end{aligned}$$

from which the estimate for the minimum follows. \square

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